5.4 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

1. Derivative of $\sin^{-1} x$

Let $y = \sin^{-1} x$, $x \in [-1, 1]$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow x = \sin y$.

Diff. w.r.t. y, we get $\frac{dx}{dy} = \cos y$.

Also $\cos y = \pm\sqrt{1-\sin^2 y}$ $\quad$ ($\because \cos^2 y + \sin^2 y = 1$)

But $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \cos y \geq 0 \Rightarrow \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$.

We know that $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, provided $\frac{dx}{dy} \neq 0$ *i.e.* $\cos y \neq 0$ *i.e.* $y \neq \pm\frac{\pi}{2}$

($\because x = \sin y$)

i.e. $x \neq \pm 1$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}, \; x \in [-1, 1] \text{ and } x \neq \pm 1.$$

Thus, $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$ *i.e.* $|x| < 1$.

2. Derivative of $\cos^{-1} x$

Let $y = \cos^{-1} x$, $x \in [-1, 1]$, $y \in [0, \pi]$ $\Rightarrow x = \cos y$.

Diff. w.r.t. y, we get $\frac{dx}{dy} = -\sin y$.

Also $\sin y = \pm\sqrt{1-\cos^2 y}$ $\quad (\because \sin^2 y + \cos^2 y = 1)$

But $y \in [0, \pi] \Rightarrow \sin y \geq 0 \Rightarrow \sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}, \; x \in [-1, 1] \text{ and } x \neq \pm 1.$$

Thus, $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$ *i.e.* $|x| < 1$.

3. Derivative of $\tan^{-1} x$

Let $y = \tan^{-1} x$, for all $x \in \mathbf{R}$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow x = \tan y$.

Diff. w.r.t. y, we get $\frac{dx}{dy} = \sec^2 y$.

But $\sec^2 y = 1 + \tan^2 y = 1 + x^2$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}, \text{ for all } x \in \mathbf{R}.$$

Thus, $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, for all $x \in \mathbf{R}$.

4. Derivative of $\cot^{-1} x$

$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$, for all $x \in \mathbf{R}$. (Proof is left for the reader as an exercise)

5. Derivative of $\sec^{-1} x$

Let $y = \sec^{-1} x$, $|x| \geq 1$, $y \in [0, \pi]$ except $\frac{\pi}{2} \Rightarrow y = \cos^{-1}\left(\frac{1}{x}\right)$.

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{\sqrt{\frac{x^2-1}{x^2}}} \cdot (-1) \cdot x^{-2}$$

$$= \frac{|x|}{\sqrt{x^2-1}} \cdot \frac{1}{x^2} = \frac{|x|}{\sqrt{x^2-1}\,|x|^2} = \frac{1}{|x|\sqrt{x^2-1}}, \; |x| > 1.$$

Thus, $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$.

Corollary. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$, $x > 1$.

REMARK

Generally, the practice is to use $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ and in such circumstances it is understood that $x > 1$.

6. Derivative of $\operatorname{cosec}^{-1} x$

$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, |x| > 1.$ (Proof is left for the reader as an exercise)

Corollary. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}, x > 1.$

REMARK

Generally, the practice is to use $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$ and in such circumstances it is understood that $x > 1$.

5.4.1 Differentiation by substitution

There are no hard and fast rules for making suitable substitutions. It is the experience which guides us for the selection of a proper substitution. However, some useful suggestions are given below :

If the function contains an expression of the form

(i) $a^2 - x^2$, put $x = a \sin t$ or $x = a \cos t$

(ii) $a^2 + x^2$, put $x = a \tan t$ or $x = a \cot t$

(iii) $x^2 - a^2$, put $x = a \sec t$ or $x = a \operatorname{cosec} t$

(iv) $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$, put $x = a \cos t$

(v) $a \cos x \pm b \sin x$, put $a = r \cos \alpha$ and $b = r \sin \alpha$, $r > 0$.

ILLUSTRATIVE EXAMPLES

Example 1. *Differentiate the following functions w.r.t. x :*

(i) $\tan^{-1}(x^2)$ (ii) $\sqrt{\sin^{-1}(x^2)}$ (iii) $\sin^{-1}(x\sqrt{x})$.

Solution. (i) Let $y = \tan^{-1}(x^2)$, differentiating w.r.t. x (by chain rule), we get

$$\frac{dy}{dx} = \frac{1}{1+(x^2)^2} \cdot \frac{d}{dx}(x^2) = \frac{1}{1+x^4} \cdot 2x = \frac{2x}{1+x^4}.$$

(ii) Let $y = \sqrt{\sin^{-1} x^2}$, differentiating w.r.t. x (by chain rule), we get

$$\frac{dy}{dx} = \frac{1}{2}(\sin^{-1} x^2)^{-1/2} \cdot \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{x}{\sqrt{1-x^4}\sqrt{\sin^{-1} x^2}}.$$

(iii) Let $y = \sin^{-1}(x\sqrt{x}) = \sin^{-1}(x^{3/2})$, diff. w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^{3/2})^2}} \cdot \frac{d}{dx}(x^{3/2}) = \frac{1}{\sqrt{1-x^3}} \cdot \frac{3}{2} x^{1/2} = \frac{3\sqrt{x}}{2\sqrt{1-x^3}}.$$

Example 2. *Differentiate the following functions w.r.t. x :*

(i) $x^2 \cos^{-1}(1-x)$ (ii) $\cot^{-1}\left(\frac{1-x}{1+x}\right)$.

Example 4. *If* $y = \dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}$, *prove that* $(1-x^2)\dfrac{dy}{dx} = x + \dfrac{y}{x}$.

Solution. Given $y = \dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}$ ...(*i*)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\sqrt{1-x^2}\left[x.\dfrac{1}{\sqrt{1-x^2}} + \sin^{-1}x.1\right] - x\sin^{-1}x.\dfrac{1}{2\sqrt{1-x^2}}(-2x)}{1-x^2}$$

$$\therefore \quad (1-x^2)\frac{dy}{dx} = x + \sqrt{1-x^2}\ \sin^{-1}x + \frac{x^2\sin^{-1}x}{\sqrt{1-x^2}}$$

$$= x + \frac{(1-x^2)\sin^{-1}x + x^2\sin^{-1}x}{\sqrt{1-x^2}}$$

$$= x + \frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad (1-x^2)\frac{dy}{dx} = x + \frac{y}{x}$$ (using (*i*))

Example 5. *Differentiate the following functions w.r.t. x :*

(*i*) $\cos^{-1}(\sin x)$

(*ii*) $\tan^{-1}\left(\dfrac{1-\cos x}{\sin x}\right)$

(*iii*) $\tan^{-1}\left(\sqrt{\dfrac{1+\cos x}{1-\cos x}}\right)$

(*iv*) $\tan^{-1}\left(\sqrt{\dfrac{1+\sin x}{1-\sin x}}\right)$.

Example 6. *Differentiate the following functions w.r.t. x :*

(i) $\tan^{-1}(\sec x + \tan x)$ *(Exemplar)* *(ii)* $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$. *(CBSE 2012)*

Solution. *(i)* Let $y = \tan^{-1}(\sec x + \tan x) = \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) = \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)$

$$= \tan^{-1}\left(\frac{1-\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)}\right) = \tan^{-1}\left(\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right) = \frac{\pi}{4}+\frac{x}{2},$$ differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}.$$

(ii) Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, put $x = \tan t$ *i.e.* $t = \tan^{-1} x$,

then $$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2 t}-1}{\tan t}\right) = \tan^{-1}\left(\frac{\sec t - 1}{\tan t}\right) = \tan^{-1}\left(\frac{\frac{1}{\cos t}-1}{\frac{\sin t}{\cos t}}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos t}{\sin t}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{t}{2}}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right) = \tan^{-1}\left(\tan\frac{t}{2}\right) = \frac{t}{2}$$

$$= \frac{1}{2}.\tan^{-1} x,$$ diff. w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{1+x^2} = \frac{1}{2(1+x^2)}.$$

Example 7. *Differentiate the following functions (by suitable substitutions) w.r.t. x :*

(i) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ (ii) $\tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)$

(iii) $\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$ (CBSE 2015) (iv) $\tan^{-1}\left(\sqrt{1+x^2}+x\right)$.

Solution. (i) Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, put $x = \tan t$ i.e. $t = \tan^{-1} x$,

then $y = \sin^{-1}\left(\frac{2\tan t}{1+\tan^2 t}\right) = \sin^{-1}(\sin 2t) = 2t = 2\tan^{-1} x$,

differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2.\frac{1}{1+x^2} = \frac{2}{1+x^2}.$$

REMARK

The above solution is valid only for $x \in [-1, 1]$. It may be noted that the given function is defined for all $x \in \mathbf{R}$. To get a more general solution, differentiate directly by using chain rule.

(ii) Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)$, put $x = \tan t$ i.e. $t = \tan^{-1} x$,

then $y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2 t}+1}{\tan t}\right) = \tan^{-1}\left(\frac{\sec t+1}{\tan t}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{\cos t}+1}{\frac{\sin t}{\cos t}}\right) = \tan^{-1}\left(\frac{1+\cos t}{\sin t}\right) = \tan^{-1}\left(\frac{2\cos^2\frac{t}{2}}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$= \tan^{-1}\left(\cot\frac{t}{2}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{t}{2}\right)\right) = \frac{\pi}{2}-\frac{t}{2} = \frac{\pi}{2}-\frac{1}{2}\tan^{-1} x,$$

differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 0 - \frac{1}{2}\cdot\frac{1}{1+x^2} = -\frac{1}{2(1+x^2)}.$$

(iii) Let $y = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) = \cos^{-1}\left(\frac{x-\frac{1}{x}}{x+\frac{1}{x}}\right) = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$,

put $x = \tan t$ i.e. $t = \tan^{-1} x$,

then $y = \cos^{-1}\left(\frac{\tan^2 t-1}{\tan^2 t+1}\right) = \cos^{-1}\left(-\frac{1-\tan^2 t}{1+\tan^2 t}\right) = \cos^{-1}(-\cos 2t)$

$= \cos^{-1}(\cos(\pi - 2t)) = \pi - 2t = \pi - 2\tan^{-1} x$,

differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 0 - 2.\frac{1}{1+x^2} = -\frac{2}{1+x^2}.$$

(iv) Let $y = \tan^{-1}\left(\sqrt{1+x^2}+x\right)$, put $x = \cot t$ i.e. $t = \cot^{-1} x$,

then $y = \tan^{-1}\left(\sqrt{1+\cot^2 t}+\cot t\right) = \tan^{-1}(\operatorname{cosec} t + \cot t)$

Example 8. *Find* $\frac{dy}{dx}$, *when*

(i) $y = \sin^{-1}\left(\frac{3\sin x + 4\cos x}{5}\right)$ (ii) $y = \sin^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{2}\right)$

(iii) $y = \tan^{-1}\left(\frac{\sqrt{a} - \sqrt{x}}{1+\sqrt{ax}}\right)$

Sol (i) $y = \sin^{-1}\left[\frac{3\sin x + 4\cos x}{5}\right] = \sin^{-1}\left[\frac{3}{5}\sin x + \frac{4}{5}\cos x\right]$

$= \sin^{-1}\left[\sin x\sqrt{1-\left(\frac{4}{5}\right)^2} + \frac{4}{5}\sqrt{1-\sin^2 x}\right]$

$= \sin^{-1}(\sin x) + \sin^{-1}\frac{4}{5}$ $\quad \because \left[\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)\right.$

$= x + \sin^{-1}\frac{4}{5}$

$\frac{dy}{dx} = 1 + 0 = 1$

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Example 9. *Differentiate the following functions :*

(*i*) $\sin^{-1}\left(\dfrac{5x+12\sqrt{1-x^2}}{13}\right)$ (*ii*) $y=\sin^{-1}\left(\dfrac{6x-4\sqrt{1-4x^2}}{5}\right)$ (CBSE 2016)

Solution. (*i*) Let $y=\sin^{-1}\left(\dfrac{5x+12\sqrt{1-x^2}}{13}\right)$, put $x=\sin t$ *i.e.* $t=\sin^{-1}x$,

then $y=\sin^{-1}\left(\dfrac{5\sin t+12\sqrt{1-\sin^2 t}}{13}\right)=\sin^{-1}\left(\dfrac{5}{13}\sin t+\dfrac{12}{13}\cos t\right)$.

Let $5=r\cos\alpha$ and $12=r\sin\alpha$

$\Rightarrow r^2(\cos^2\alpha+\sin^2\alpha)=5^2+12^2 \Rightarrow r^2=169 \Rightarrow r=13$

and $\tan\alpha=\dfrac{12}{5} \Rightarrow \alpha=\tan^{-1}\dfrac{12}{5}$.

$$\therefore \quad y=\sin^{-1}\left(\frac{r\cos\alpha\sin t+r\sin\alpha\cos t}{13}\right)=\sin^{-1}\left(\frac{r}{13}\sin(t+\alpha)\right)$$

$$=\sin^{-1}\left(\frac{13}{13}\sin(t+\alpha)\right)$$

$$=\sin^{-1}(\sin(t+\alpha))=t+\alpha=\sin^{-1}x+\tan^{-1}\left(\frac{12}{5}\right).$$

Diff. w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+0=\frac{1}{\sqrt{1-x^2}}.$$

(*ii*) Let $y=\sin^{-1}\left(\dfrac{6x-4\sqrt{1-4x^2}}{5}\right)=\sin^{-1}\left(\dfrac{3(2x)-4\sqrt{1-(2x)^2}}{5}\right)$

put $2x=\sin t \Rightarrow t=\sin^{-1}2x$

then $y=\sin^{-1}\left(\dfrac{3\sin t-4\sqrt{1-\sin^2 t}}{5}\right)=\sin^{-1}\left(\dfrac{3\sin t-4\cos t}{5}\right)$

Let $3=r\cos\alpha$ and $4=r\sin\alpha$

$\Rightarrow r^2(\cos^2\alpha+\sin^2\alpha)=3^2+4^2 \Rightarrow r^2=25 \Rightarrow r=5$

and $\tan\alpha=\dfrac{4}{3} \Rightarrow \alpha=\tan^{-1}\dfrac{4}{3}$,

$$\therefore \quad y=\sin^{-1}\left(\frac{r\cos\alpha\sin t-r\sin\alpha\cos t}{5}\right)=\sin^{-1}\left(\frac{r}{5}\sin(t-\alpha)\right)$$

$$=\sin^{-1}\left(\frac{5}{5}\sin(t-\alpha)\right)$$

$$=\sin^{-1}(\sin(t-\alpha))=t-\alpha$$

$$=\sin^{-1}2x-\tan^{-1}\left(\frac{4}{3}\right).$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-(2x)^2}}.2.1-0=\frac{2}{\sqrt{1-4x^2}}.$$

Example 10. (*i*) *If* $y=\sin\left(2\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}\right)$, *prove that* $\dfrac{dy}{dx}=-\dfrac{x}{\sqrt{1-x^2}}$.

(*ii*) *If* $y=\sin^2\left(\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}\right)$, *find* $\dfrac{dy}{dx}$.

Solution. (*i*) Let $x = \cos t$, we get

$$y = \sin\left(2\tan^{-1}\sqrt{\frac{1-\cos t}{1+\cos t}}\right) = \sin\left(2\tan^{-1}\sqrt{\frac{2\sin^2\frac{t}{2}}{2\cos^2\frac{t}{2}}}\right)$$

$$= \sin\left(2\tan^{-1}\left(\tan\frac{t}{2}\right)\right) = \sin\left(2.\frac{t}{2}\right) = \sin t$$

$$= \sqrt{1-\cos^2 t} = \sqrt{1-x^2}$$, differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2}\,(0-2x) = -\frac{x}{\sqrt{1-x^2}}.$$

(*ii*) Let $x = \cos t$, we get

$$y = \sin^2\left(\tan^{-1}\sqrt{\frac{1-\cos t}{1+\cos t}}\right) = \sin^2\left(\tan^{-1}\sqrt{\frac{2\sin^2\frac{t}{2}}{2\cos^2\frac{t}{2}}}\right) = \sin^2\left(\tan^{-1}\left(\tan\frac{t}{2}\right)\right)$$

$$= \sin^2\frac{t}{2} = \frac{1-\cos t}{2} \qquad \left(\because 1-\cos t = 2\sin^2\frac{t}{2}\right)$$

$$= \frac{1}{2}(1-x)$$, diff. w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}(0-1) = -\frac{1}{2}.$$

Example 11. *Differentiate the following functions w.r.t. x :*

(*i*) $\tan^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$ (CBSE 2013)

(*ii*) $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$. (CBSE 2015)

Solution. (*i*) Let $y = \tan^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$$= \tan^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\times\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right)$$

$$= \tan^{-1}\left(\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})^2}{(1+\sin x)-(1-\sin x)}\right)$$

$$= \tan^{-1}\left(\frac{(1+\sin x)+(1-\sin x)+2\sqrt{1+\sin x}\,\sqrt{1-\sin x}}{2\sin x}\right)$$

$$= \tan^{-1}\left(\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}\right) = \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right) = \tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\cot\frac{x}{2}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right)$$

$$= \frac{\pi}{2}-\frac{x}{2}$$, diff. w.r.t. x, we get

$$\frac{dy}{dx} = 0-\frac{1}{2}.1 = -\frac{1}{2}.$$

Example 12. *If* $y = \tan^{-1}\left(\frac{5x}{1-6x^2}\right)$, $-\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}}$, *then prove that* $\frac{dy}{dx} = \frac{2}{1+4x^2} + \frac{3}{1+9x^2}$.

Solution. Given $y = \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \tan^{-1}\left(\frac{2x+3x}{1-2x.3x}\right)$

$$\left(-\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}} \Rightarrow |x| < \frac{1}{\sqrt{6}} \Rightarrow x^2 < \frac{1}{6} \Rightarrow 2x.3x < 1\right)$$

$$= \tan^{-1} 2x + \tan^{-1} 3x.$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+(2x)^2}.2.1 + \frac{1}{1+(3x)^2}.3.1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{1+4x^2} + \frac{3}{1+9x^2}.$$

Example 15. *If* $y = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right)$, *prove that* $\frac{dy}{dx} = \frac{1}{a + b\cos x}$, $a > b > 0$.

Solution. Given $y = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right)$, diff. w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2}{\sqrt{a^2 - b^2}} \cdot \frac{1}{1 + \left(\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right)^2} \cdot \sqrt{\frac{a-b}{a+b}} \cdot \sec^2\frac{x}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{a-b}}{\sqrt{a+b}\sqrt{a^2 - b^2}} \cdot \frac{(a+b)\sec^2\frac{x}{2}}{(a+b) + (a-b)\tan^2\frac{x}{2}} = \frac{1}{(a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2}}$$

$$= \frac{1}{a\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) + b\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)} = \frac{1}{a + b\cos x}.$$

Example 17. *If* $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$*, prove that* $\frac{dy}{dx}$ *is independent of* x*.*

Solution. Given $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$ $\left(\because \sec^{-1}x = \cos^{-1}\frac{1}{x}\right)$

$= \cos^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$ $\left(\because \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}\right)$

$= \frac{\pi}{2}$

$\therefore$ $\frac{dy}{dx} = 0$, which is independent of x.

Derivative of Inverse Trigonometric Functions

Questions for Practice

Differentiate the following w.r.t x.

1. $\cos^{-1}(2x\sqrt{1-x^2})$
2. $\sec^{-1}\left(\frac{1}{4x^3-3x}\right)$
3. $\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$
4. $\sin^{-1}\left(\frac{x}{\sqrt{x^2+a^2}}\right)$
5. $\sin^{-2}\left(\cot^{-1}\left(\sqrt{\frac{1+x}{1-x}}\right)\right)$
6. $\tan^{-1}\left(\frac{4\sqrt{x}}{1-4x}\right)$
7. $\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$
8. If $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$
   Prove that $\frac{dy}{dx}$ is independent of x.
9. If $y = 2\cos^{-1}(\sin x) + 3\cot^{-1}(\tan x)$ find $\frac{dy}{dx}$.
10. If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, Prove that
    $(1+x^2)\frac{dy}{dx} = 2$.

Q11 Diff. $\sin(2\sin^{-1}x)$

Q12 Diff. $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$